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Chaotic dynamics of an air-damped bouncing ball

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A ball bouncing elastically upon a vertically vibrated platform is one of the simplest examples of a chaotic system. If dissipation is introduced at each bounce through a coefficient of restitution, the motion is no longer chaotic; the trajectories exhibit locking solutions that result in periodic behavior. Here we investigate the dynamics of a bouncing ball influenced by air damping. We consider the effects of both static air and air moving with the platform, and show that there is an exact mapping between them. In either case, the system has a rather complex dynamical behavior including truly chaotic trajectories. Our results highlight the importance of air effects for fine particulate systems.

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One of the simplest examples of a nonlinear dynamical system is a ball bouncing elastically upon a vertically vibrated surface [1]. Such a system exhibits a range of complex behavior including periodic motion, a period-doubling cascade, and fully developed chaos, depending upon the amplitude and frequency of the surface vibration [2]. This idealized system captures many of the features observed experimentally [3–7]. However, in reality, the ball undergoes inelastic collisions with the surface, dissipating energy. If this energy loss is accounted for by introducing a constant coefficient of restitution $r<1$, the resulting system is no longer truly chaotic; it exhibits locking solutions [8,9]. The relative velocity of the ball and the surface goes to zero within a large but finite time; the ball will then leave the surface again when the downward acceleration exceeds g and this behavior will repeat periodically [10].

The inelastic bouncing ball has been used as a simplified model of vibrated granular media [11]. Recently, however, it has been realized that for fine particulates, the interaction of the particles with the surrounding air can have important effects on the dynamical behavior [12]. Specifically, it has been shown that air effects are the driving mechanism for Faraday piling [13], the air-driven Brazil nut effect [14], and separation in a mixture of fine particulates [15].

In this note, we consider the dynamical behavior of a single air-damped bouncing ball. It is well known that, if Stokes's law damping is the only dissipative mechanism, a ball dropped onto a stationary surface does not come to rest in a finite time. This is because the dissipation goes to zero linearly with the ball's velocity. Here we investigate the behavior of the ball if the surface is vibrated vertically. We consider two situations; either the air is assumed to move with the surface or to remain static. We find that the dynamics exhibit many features in common with the elastic ball problem, in particular, the ball never comes to rest on the surface and the motion is thus truly chaotic.

Consider a single particle of mass m moving vertically above a sinusoidally vibrating surface under the influence of gravity and air damping. We will initially assume that the air is moving with the surface and influences the ball through Stokes's law drag. The dynamics of the ball can also be considered to be one dimensional. The vertical position of the ball, z, and the surface, z_s, obey the equations

$$m\frac{d^2z}{dt^2}+\mu\left(\frac{dz}{dt}-\frac{dz_s}{dt}\right)+mg=0 \tag{1}$$

and

$$z_s=A\sin(\omega t), \tag{2}$$

respectively. Here, μ is the viscous drag coefficient, A is the amplitude of vibration, and ω is the vibration frequency. It is convenient to introduce dimensionless variables

$$y=\frac{z}{A},\qquad \tau=\omega t \tag{3}$$

for the position of the particle, z, at time t, and

$$\Delta=\frac{\mu}{m\omega},\qquad \Gamma=\frac{A\omega^2}{g}, \tag{4}$$

which characterize the viscous dissipation and acceleration of the surface, respectively.

Equations (1) and (2) can readily be solved analytically for any initial conditions. Assume that at some (rescaled) time τ_0, the particle leaves the surface at height y_0 with vertical velocity v_0. The corresponding height of the particle and surface after a time interval τ are given by

$$\begin{aligned}y(\tau)=y_0-\frac{\tau}{\Gamma\Delta}+\frac{1}{\Delta}\left(v_0+\frac{1}{\Gamma\Delta}\right)(1-e^{-\Delta\tau})\\+\frac{1}{\Delta}[\alpha\sin(\tau_0)-\beta\cos(\tau_0)](1-e^{-\Delta\tau})\\+\alpha[\cos(\tau+\tau_0)-\cos(\tau_0)]+\beta[\sin(\tau+\tau_0)-\sin(\tau_0)]\end{aligned} \tag{5}$$

and

$$y_s(\tau)=\sin(\tau+\tau_0), \tag{6}$$

where

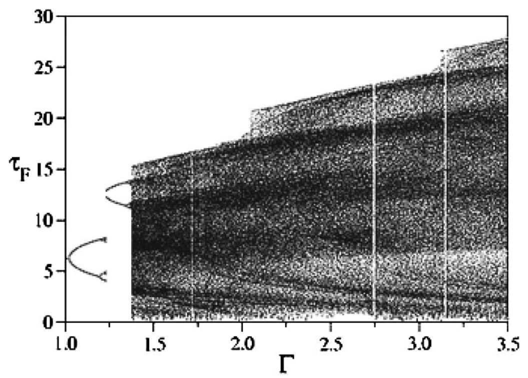

FIG. 1. Bifurcation diagram for $\Delta = 0.05$ plotted in the space of τ_F vs Γ. At each value of Γ, the system is relaxed for 10 000 bounces and the last 256 points are plotted. Γ is increased by 0.01 between measurements.

$$\alpha = -\frac{\Delta}{1+\Delta^2}, \quad \beta = \frac{\Delta^2}{1+\Delta^2}. \tag{7}$$

The ball next collides with the surface when

$$y(\tau_F) = y_s(\tau_F), \tag{8}$$

which defines the time of flight τ_F. If the collision is assumed to be elastic, the ball will then leave the surface at $y_s(\tau_F)$ with a velocity

$$v = 2\frac{dy_s}{d\tau}\bigg|_{\tau_F} - \frac{dy}{d\tau}\bigg|_{\tau_F}. \tag{9}$$

We have used the above equations to determine the motion of the ball numerically. For given initial conditions, Eq. (8) can be solved to find the time of flight until the next collision. Equation (9) can then be used to calculate the new launch velocity, which acts as the input for the following interval of the motion. Repeating this procedure, one can build up the exact trajectory of the ball for all subsequent times.

Figure 1 shows a plot of the time of flight τ_F against Γ. We choose to plot τ_F rather than the phase of the surface at the collision because it contains important information about the particle's actual trajectory. The plot was generated by relaxing the system for 10 000 bounces at each Γ and recording the last 256 flight times. After each measurement, Γ was increased by a small amount, which mimics the experimental method that would be used to determine such a plot [4]. The value of $\Delta = 0.05$ corresponds approximately to a bronze sphere of diameter 0.5 mm in the presence of air.

We first consider the dynamics of a ball, which initially is at rest on the surface. For Γ slightly greater than 1, the ball exhibits periodic motion with the period of the vibrating surface, $\tau_F = 2\pi$. As Γ increases, there are period-doubling bifurcations around this period-1 solution. However, for $\Gamma \approx 1.23$, the motion jumps to a two cycle centered on the period-2 solution, $\tau_F = 4\pi$. On increasing Γ further, there is a second bifurcation cascade up to $\Gamma \approx 1.38$, beyond which the system exhibits chaotic behavior. Even within the chaotic

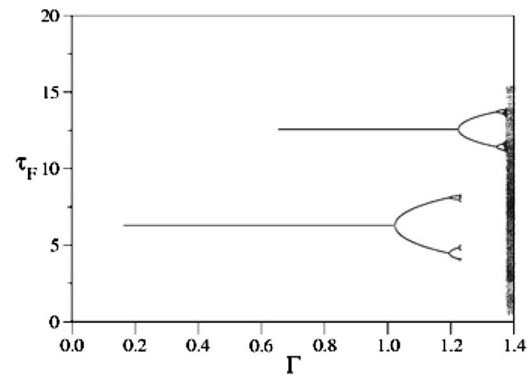

FIG. 2. Bifurcation diagram for $\Delta = 0.05$ obtained by decreasing Γ. Two different sets of data are shown. The upper branch was obtained starting at $\Gamma = 1.4$ (within the chaotic region) and the lower branch starting at $\Gamma = 1.23$. The measured values of τ_F are very sensitive to the initial conditions and to the way in which Γ is varied.

regime, there are apparent discontinuous changes in the measured flight time as Γ is varied. They are close to values of τ_F that are integer multiples of the driving period, $\tau_F = 2n\pi$.

To understand the existence of the discontinuous jumps between seemingly different solutions in Fig. 1, we have considered the behavior of the ball as Γ is reduced. This is shown in Fig. 2. We have started from values of Γ chosen to illustrate specific behaviors. The detailed structure depends very sensitively upon the way in which Γ is reduced, for example, during which part of the motion it is changed. The ball can remain in a period-2 cycle even for $\Gamma < 1$, or it can revert to period-1 behavior as observed earlier. Similarly, higher order branches can be accessed if the ball is given a large initial velocity, or if it is dropped onto the surface. For each of the single-period branches with flight times $\tau_F = 2n\pi$, a linear stability analysis shows that the first bifurcation occurs when

$$\Gamma = (1+\Delta^2)^{1/2}\left(x^2 + \frac{1}{\Delta^2}(1-x)^2\right)^{1/2}, \tag{10}$$

where $x = n\pi\Delta \coth(n\pi\Delta)$. For large n, this value of Γ varies linearly with n. The complex behavior shown in Fig. 1 results from a superposition of the different stable branches of the bifurcation diagram. Which of the trajectories the ball takes depends upon the initial conditions and the way in which Γ is varied. The system thus exhibits extremely hysteretic behavior.

For the range of Γ that we have considered, we note that there are very few values of $\tau_F < 0.1$. This suggests that the ball does not undergo a rapid sequence of collisions resulting in a locking solution. To test this further, we have performed long numerical calculations of the trajectories for $\Gamma = 2$. We find that the ball remains in motion even for (unscaled) times corresponding to several months. While no numerical work can rule out the existence of locking solutions, our results are highly suggestive that the ball with air damping alone exhibits truly chaotic dynamics.

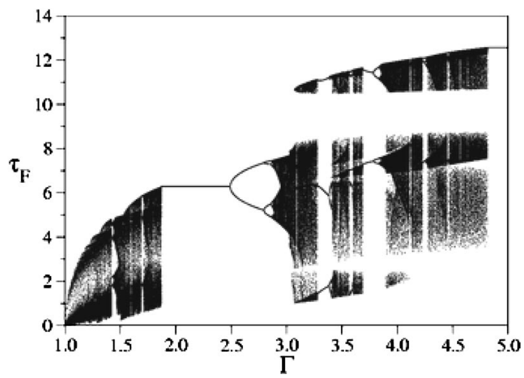

FIG. 3. Bifurcation diagram for $\Delta = 0.5$ obtained by increasing Γ from 1.

The bifurcation diagram for a higher value of the damping parameter Δ is shown in Fig. 3. It exhibits a much more complex structure than in the weakly damped case. Along with the period-doubling transitions leading to chaotic behavior, we find values of Γ where there are three cycles, and regions where chaos gives way to simple periodic motion. In general, increasing Δ tends to extend the region of stability of the single-period motion to higher values of Γ, as described by Eq. (10). Thus, the period-doubling cascades associated with each of the $\tau_F = 2n\pi$ solutions are more apparent. For the range of parameters that we have investigated, and the times that are accessible by our numerical method, we find no evidence that the ball comes to rest on the surface. Thus, even for high values of the damping parameter, the motion remains chaotic.

The dynamics described by Eq. (1) assumes that the air is moving with the surface. If the surface is constructed to be permeable to air [14], the air above the surface will remain largely unaffected by the surface's motion. The corresponding equations governing the ball's motion are the same as in Eqs. (6) and (7) but with α and β both equal to zero. By making the change of variable $u_0 = (1+\Delta^2)^{1/2}[v_0 + \alpha \sin(\tau_0) - \beta \cos(\tau_0)]$, the recursion relation for τ_F in the moving air problem maps exactly onto that for the static air problem. The only difference is that Γ is rescaled by a factor $(1+\Delta^2)^{1/2}$, i.e.,

$$\Gamma_m = (1+\Delta^2)^{1/2}\Gamma_s, \tag{11}$$

where the subscripts m and s refer to the moving and static air problems, respectively. Thus, the bifurcation diagrams for the static air problem have exactly the same structure as described above for the moving air problem. Note, that in order to achieve the same flight time, Γ_m must be greater than Γ_s. Thus, for a given Γ, a ball damped by static air will be more active than a ball influenced by moving air. This has important consequences for fine granular materials vibrated in porous containers.

It is clear that the model that we have considered represents a rather idealized treatment of an air-damped bouncing ball. However, given the simplicity of the model, the system exhibits rather rich and complex dynamical behavior. Our findings show that the coupling between the particle's motion and the motion of the air induced by vibration can have significant effects on the dynamics of the system.

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